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Radiation of a Hertzian oscillator moving with superluminal velocity through a dielectric medium

T. P. KHAN

Dinabandhu Andrews College, Garia, 24 Parganas, West Bengal, India

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Abstract. The radiation of a Hertzian oscillator (excited atom) moving through a dielectric medium with a superluminal velocity has been studied with the purpose of its application as a possible source of generating microwaves. Three types of radiation occur: the frequencies of the emitted radiation have an angular dependence similar to that in the normal Doppler effect, the anomalous Doppler effect and Čerenkov radiation respectively. Radiation is observed not only at the Čerenkov angle, but also both outside and inside the Čerenkov cone.

1. Introduction

When an electron moves through a dielectric medium with a velocity greater than the phase velocity of light we obtain Čerenkov radiation which is usually observed in the optical region. There is no limitation to the Čerenkov radiation on the long-wave side provided the medium is free from absorption bands in this region of spectrum, and Ginsburg was the first to propose that the Čerenkov radiation could in principle be used to produce microwaves in the range $\lambda \sim 0{\cdot}01$ to $1{\cdot}0$ cm (Jelley 1958). However, since the yield of the radiation falls with decreasing frequency, it is difficult to obtain high powers at microwave frequences. Moreover, such radiation would give a continuous spectrum, whereas in most applications of microwaves we are interested in a single-frequency oscillation. Such difficulties may be largely overcome with the Čerenkov radiation of excited atoms when the radiation tends to concentrate in a narrow band of frequencies.

Current interest lies in the Čerenkov radiation given by a system of particles moving through a dielectric medium. A system of particles may have a natural frequency of oscillation. Therefore, an examination of the behaviour of an oscillator of arbitrary natural frequency moving in a medium may be of practical as well as theoretical interest (Frank 1959). The search for ways of generating microwaves has produced an abundance of ideas, including the direct Čerenkov radiation from electrons moving near a dielectric (Danos 1955), and the 'undulator' scheme proposed by Motz (1951). Classical treatments of the problem of Čerenkov radiation from moving dipoles and moving oscillators have been given by Balazs (1956), Ginsburg and Eidman (1959) and Ginsburg (1960).

The purpose of this communication is to investigate the problem of the radiation of a Hertzian oscillator from the point of view of its application in generating microwaves. Contrary to the situation with Čerenkov radiation from a single charge (electron), three types of radiation are found. The frequencies of emission have the angular dependence similar to those of Čerenkov radiation, the normal and the anomalous Doppler effect. The radiation not only occurs at a particular angle on the surface of a cone but is emitted both outside and inside the Čerenkov cone. The radiation output per unit angular frequency is found to have a maximum at a frequency close to the frequency of the oscillator. If in the present analysis the frequency of the oscillator is made equal to zero, we obtain the expression for energy radiation obtained by Balazs for the case of a fixed dipole.

2. Statement of the problem

The Hertzian oscillator (Sommerfeld 1952) is considered to consist of a moving charge $+q$ with a neighbouring charge $-q$ at a distance l of separation which varies with time. Thus we consider the separation length $l = l_0 \cos \omega_0 t$. We choose a coordinate system fixed in an isotropic medium of dielectric constant ϵ. The z axis represents the direction of motion. The dipole axis is always assumed to be oriented in a direction parallel to the z axis. We assume that the Čerenkov condition is fulfilled (i.e. v is greater than the phase velocity of light in the medium).

3. Equations and method of solution

The charge and current densities of an oscillator moving with velocity v in a dielectric medium with its axis parallel to the direction of motion are

$$\begin{aligned} \rho(x, t) &= q\delta(x)\delta(y)\{\delta(z - l_0 \cos \omega_0 t - vt) - \delta(z - vt)\} \\ J_z(x, t) &= v\rho(x, t) - ql_0\omega_0 \sin \omega_0 t\delta(x)\delta(y)\delta(z - l_0 \cos \omega_0 t - vt) \\ J_x &= J_y = 0. \end{aligned} \tag{1}$$

The Fourier transforms of $\rho(x, t)$ and $J(x, t)$ are, according to the general rule of transformation (Jackson 1962),

$$F(x, t) = \frac{1}{(2\pi)^2} \int \mathrm{d}^3k \int \mathrm{d}\omega F(k, \omega) \exp(\mathrm{i}k \, . \, x - \mathrm{i}\omega t) \tag{2}$$

obtained as follows:

$$\begin{aligned} \rho(k, \omega) &= \frac{\mathrm{i}k_z P}{4\pi} \{\delta(\omega - \omega_0 - k \, . \, v) + \delta(\omega + \omega_0 - k \, . \, v)\} \\ J_z(k, \omega) &= \frac{\mathrm{i}P}{4\pi} \{(k_z v - \omega_0)\delta(\omega - \omega_0 - k \, . \, v) + (k_z v + \omega_0)\delta(\omega + \omega_0 - kv)\} \end{aligned} \tag{3}$$

where

$$P = l_0 q \text{ is the dipole moment.}$$

When $v = 0$ we have the case of a stationary oscillator. The Fourier transforms of the potential are:

$$\begin{aligned} \phi(\omega) &= \frac{\mathrm{i}(\omega - \omega_0)P}{\epsilon v^2(2\pi)^{1/2}} \{\mathrm{K}_0(\lambda_1\rho) + \mathrm{i}\pi \mathrm{I}_0(\lambda_1\rho)\} \exp\left\{-\mathrm{i}(\omega - \omega_0)\frac{z}{v}\right\} \\ &\quad + \frac{\mathrm{i}(\omega + \omega_0)P}{\epsilon v^2(2\pi)^{1/2}} \{\mathrm{K}_0(\lambda_2\rho) + \mathrm{i}\pi \mathrm{I}_0(\lambda_2\rho)\} \exp\left\{-\mathrm{i}(\omega + \omega_0)\frac{z}{v}\right\} \\ A_z(\omega) &= \frac{\mathrm{i}(\omega - 2\omega_0)P}{(2\pi)^{1/2}cv} \{\mathrm{K}_0(\lambda_1\rho) + \mathrm{i}\pi \mathrm{I}_0(\lambda_1\rho)\} \exp\left\{-\mathrm{i}(\omega - \omega_0)\frac{z}{v}\right\} \\ &\quad + \frac{\mathrm{i}(\omega + 2\omega_0)P}{(2\pi)^{1/2}cv} \{\mathrm{K}_0(\lambda_2\rho) + \mathrm{i}\pi \mathrm{I}_0(\lambda_2\rho)\} \exp\left\{-\mathrm{i}(\omega + \omega_0)\frac{z}{v}\right\}. \end{aligned} \tag{4}$$

The condition of radiation will be obtained from the argument of the δ-function

(Ginsburg and Eidman 1959) in the above equation:

$$\omega = \frac{\omega_0}{1-n\beta\cos\theta} \qquad n\beta\cos\theta < 1 \tag{5a}$$

$$\omega = \frac{\omega_0}{n\beta\cos\theta-1} \qquad n\beta\cos\theta > 1 \tag{5b}$$

$$\omega \to \infty \qquad n\beta\cos\theta = 1 \tag{5c}$$

where n is the refractive index and θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{v}$. These formulae can also be deduced from the laws of conservation of energy and momentum (see Frank 1959). When $n\beta < 1$, the frequency $\omega(\theta)$ diminishes as θ increases; this effect is termed the normal Doppler effect. However, if $n\beta > 1$, the radiation must be considered both outside and inside the Čerenkov cone. Inside the Čerenkov cone ($\theta < \theta_0$, $\cos\theta_0 = 1/n\beta$) the Doppler effect is called anomalous and the frequency $\omega(\theta)$ increases as θ increases. If $n(\omega)$ is constant, ω attains an infinitely large value as θ tends to θ_0. Outside the Čerenkov cone the frequency $\omega(\theta)$ diminishes as θ increases. We find equations (5) give three characteristic modes of radiation. Under the condition:

(5*a*) $v\cos\theta < \dfrac{c}{n}$ we have emission of radiation as normal Doppler effect

(5*b*) $v\cos\theta > \dfrac{c}{n}$ we have emission of radiation as anomalous Doppler effect

(5*c*) $v\cos\theta = \dfrac{c}{n}$ we have emission of Čerenkov radiation.

These conditions of radiation are deducible from the necessary and sufficient condition

$$v\cos\theta = W \tag{6}$$

where W is the group velocity of light and

$$\frac{1}{W} = \frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{n}{c} + \frac{\omega}{c}\frac{\mathrm{d}n}{\mathrm{d}\omega} \tag{7}$$

where

$$n^2(\omega) = 1+\alpha\sum_i \frac{f_i}{\omega_i^2-\omega^2-\mathrm{i}g_i\omega} \qquad \text{(Jelley 1958, p. 53).}$$

$n(\omega)$ is thus a complicated function of ω. Because of the term $\mathrm{d}n/\mathrm{d}w$, which may assume positive or negative values (including zero), we have three conditions for the radiation of a Hertzian oscillator moving through a dielectric medium with superluminal velocity. The necessary condition (although not sufficient) of Čerenkov radiation of a single charge moving with velocity v through a dielectric medium is the condition (5*c*) as stated above.

From the definition of electromagnetic fields in terms of potentials, we obtain their Fourier transforms (Morse and Feshbach 1953) in cylindrical polar coordinates

(ρ, ϕ, z) at a point remote from the particle track $(\rho \to \infty)$ as follows:

$$E_\rho(\omega) = \frac{\mathrm{i}(\omega-\omega_0)P\lambda_1}{(2\pi)^{1/2}\epsilon v^2}\{\mathrm{K}_1(\lambda_1\rho)+\mathrm{i}\pi\mathrm{I}_1(\lambda_1\rho)\}\exp\left\{-\mathrm{i}(\omega-\omega_0)\frac{z}{v}\right\} + \frac{\mathrm{i}(\omega+\omega_0)P\lambda_2}{(2\pi)^{1/2}\epsilon v^2}\{\mathrm{K}_1(\lambda_2\rho)+\mathrm{i}\pi\mathrm{I}_1(\lambda_2\rho)\}\exp\left\{-\mathrm{i}(\omega+\omega_0)\frac{z}{v}\right\} \tag{8a}$$

$$E_z(\omega) = \frac{\omega(\omega-2\omega_0)P}{(2\pi)^{1/2}c^2v}\left\{1-\frac{1}{n^2\beta^2}\frac{(\omega-\omega_0)^2}{\omega(\omega-2\omega_0)}\right\}\{\mathrm{K}_0(\lambda_1\rho)+\mathrm{i}\pi\mathrm{I}_0(\lambda_1\rho)\}\exp\left\{-\mathrm{i}(\omega-\omega_0)\frac{z}{v}\right\} + \frac{\omega(\omega+2\omega_0)P}{(2\pi)^{1/2}c^2v}\left\{1-\frac{1}{n^2\beta^2}\frac{(\omega+\omega_0)^2}{\omega(\omega+2\omega_0)}\right\}\{\mathrm{K}_0(\lambda_2\rho)+\mathrm{i}\pi\mathrm{I}_0(\lambda_2\rho)\}\exp\left\{-\mathrm{i}(\omega+\omega_0)\frac{z}{v}\right\} \tag{8b}$$

$$H_\phi(\omega) = \frac{-\mathrm{i}\lambda_1(\omega-2\omega_0)P}{(2\pi)^{1/2}cv}\{\mathrm{K}_1(\lambda_1\rho)+\mathrm{i}\pi\mathrm{I}_1(\lambda_1\rho)\}\exp\left\{-\mathrm{i}(\omega-\omega_0)\frac{z}{v}\right\} - \frac{\mathrm{i}\lambda_2(\omega+2\omega_0)P}{(2\pi)^{1/2}cv}\{\mathrm{K}_1(\lambda_2\rho)+\mathrm{i}\pi\mathrm{I}_1(\lambda_2\rho)\}\exp\left\{-\mathrm{i}(\omega+\omega_0)\frac{z}{v}\right\} \tag{8c}$$

where K_1, K_0 and I_1, I_0 are modified Hankel and Bessel functions.

The quantities λ_1 and λ_2 are imaginary, as can be shown from (5*a*) and (5*b*), because

$$\lambda_1^2 = \frac{(\omega-\omega_0)^2}{v^2}-\frac{\omega^2\epsilon(\omega)}{c^2}$$
$$\lambda_2^2 = \frac{(\omega+\omega_0)^2}{v^2}-\frac{\omega^2\epsilon(\omega)}{c^2}. \tag{9}$$

The fields characterized by subscripts 1 and 2, refer to the fields of normal Doppler and anomalous Doppler effect respectively. From (5) it can be shown that they are respectively outside and inside the Čerenkov cone. We obtain the expression for energy radiated by the oscillator through the surface of a cylinder whose axis coincides with the line of motion of the particle track per unit length:

$$\frac{\mathrm{d}I(\omega)}{\mathrm{d}l} = \frac{P^2(1-\beta^2)}{c^2v}\int_0^\infty \omega\left\{(\omega^2+4\omega_0^2)\left(1-\frac{1}{n^2\beta^2}\right)-\frac{\omega_0^2}{n^2\beta^2}\right\}\mathrm{d}\omega. \tag{10}$$

It may be mentioned that for a practical case we have a fast-moving excited atom having a typical electric dipole moment P of say 3×10^{-18} (esu × cm) travelling at a velocity $v = 2\times10^{10}$ cm s^{-1} which can cross a thin Lucite sheet in a time comparable with the de-excitation time, and so is able to emit its characteristic frequency of radiation ω_0 in a transition from a higher energy state to a lower energy state. We find that the radiation output per unit angular frequency is, from equation (10),

$$\frac{\mathrm{d}I(\omega)}{\mathrm{d}\omega\,\mathrm{d}l} = F(\omega) = \frac{P^2(1-\beta^2)}{c^2v}\omega\left\{(\omega^2+4\omega_0^2)\left(1-\frac{1}{n^2\beta^2}\right)-\frac{\omega_0^2}{n^2\beta^2}\right\}. \tag{11}$$

$F(\omega)$ is found to be maximum at a particular frequency of emission ω_m given by

$$3\omega_\mathrm{m}^2 = \frac{\omega_0^2}{n^2\beta^2-1}-4\omega_0^2. \tag{12}$$

For a practical case (Jelley 1958, p. 144)

$$n = 1{\cdot}5, \quad \beta = 0{\cdot}7, \quad \text{so } \omega_\mathrm{m}^2 = 2\omega_0^2.$$

The value of ω_m is obtained by differentiating $F(\omega)$ with respect to ω and equating to zero. Thus we find that the radiation output can be concentrated on a particular frequency close to ω_0.

We also find that when $\omega_0 = 0$, the expression for energy radiation per unit length (equation (10)) is

$$\frac{\mathrm{d}I(\omega)}{\mathrm{d}l} = \frac{P^2(1-\beta^2)}{c^2 v} \int_0^{\alpha} \omega^3 \left(1 - \frac{1}{n^2\beta^2}\right) \mathrm{d}\omega. \tag{13}$$

This is the expression for a fixed dipole and is exactly similar to that of Balazs (Jelley 1958, p. 33). The factor $1-\beta^2$ arises from the Lorentz contraction of the dipole length.

The dipole radiation will be obtained if we put $v = 0$ in equation (1) and begin the problem from this point *ab initio*. The Hertz potential is obtained from equation (4) by the same technique of Fourier transforms in spherical coordinates as

$$\phi(\omega) = \frac{P\cos\theta}{k\rho} \exp(-ik\,.\,\rho). \tag{14}$$

The frequency of emission of this radiation will always be that of the oscillator itself and is evident from the argument of the δ-function.

4. Conclusions

In the present analysis it is found that, in the case of an oscillator moving through a dielectric medium, there will be emission of radiation at velocities v which may be greater than or smaller than the c/n phase velocity of light. When $v < c/n$ there will be the normal Doppler effect while for $v > c/n$ the anomalous Doppler effect will occur. Unlike Čerenkov radiation it will be visible not only on the surface of a cone but also inside and outside the cone, satisfying three conditions of radiation as stated before. We strongly believe that radiation yield (equation (10)) will be detectable. The calculation suggests that this effect may be considered as a method of generation of radiation in the region of those frequencies which are not easily covered by other methods.

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References

Balazs, N. L., 1956, *Phys. Rev.*, **104,** 1220–2.

Danos, M., 1955, *J. appl. Phys.*, **26,** 2–7.

Frank, I. M., 1959, *Sov. Phys.—JETP*, **9,** 580–6.

Ginsburg, V. L., 1960, *Sov. Phys. Usp.*, **2,** 874–93.

Ginsburg, V. L., and Eidman, V., Ya, 1959, *Sov. Phys.—JETP*, **9,** 1300–7.

Jackson, J. D., 1962, *Classical Electrodynamics* (New York: J. Wiley), p. 550.

Jelley, J. W., 1958, *Čerenkov radiation and its applications* (Oxford: Pergamon Press), p. 257.

Morse, P. M., and Feshbach, H., 1953, *Methods of Theoretical physics*, Vols 1 and 2 (New York: McGraw-Hill), pp. 621, 827, 1325, 1362.

Motz, H., 1951, *J. appl. Phys.*, **22,** 527–35.

Sommerfeld, A., 1952, *Electrodynamics*, Vol. 3 (New York: Academic Press), pp. 148–9.